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T–S model-based nonlinear moving-horizon $\mathcal{H}_\infty$ control and applications

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Abstract

Using a T–S fuzzy model, we propose a formulation for a moving horizon $\mathcal{H}_\infty$ control problem for nonlinear systems. Our main contribution is combination of T–S fuzzy models and a moving horizon $\mathcal{H}_\infty$ control scheme by Chen and Scherer to address the disturbance attenuation of nonlinear constrained system. A sufficient condition for the disturbance amplitude is given to guarantee the feasibility of the optimization problem at each time. A parameter-dependent state feedback control law is adopted, and the corresponding optimization problem is reduced to a convex optimization problem involving linear matrix inequalities (LMIs). The $\mathcal{H}_\infty$ attenuation index for the nonlinear moving horizon $\mathcal{H}_\infty$ control problem is adapted online to satisfy time-domain constraints. Finally, the effectiveness of the proposed scheme is successfully demonstrated by control of a continuous stirred tank reactor and active queue management for routers.

1. Introduction

In recent years, Takagi–Sugeno (T–S) fuzzy control [1] has become one of the most popular and promising approaches in model-based fuzzy control because it can combine the flexibility of fuzzy logic theory and the rigorous mathematical theory of linear or nonlinear system in a unified framework. Numerous successful applications of fuzzy control have sparked analysis and design of fuzzy control systems described by a set of If–Then rules [2].

There are many results in the literature on stability analysis and design. For continuous-time T–S fuzzy control systems, three different types of Lyapunov function have been used for stability analysis and fuzzy controller synthesis: quadratic, nonquadratic, and piecewise. Among these, nonquadratic Lyapunov functions (NQLFs), also called fuzzy Lyapunov functions, usually allow more relaxed stability conditions [3–5]. However, the problem of finding the least conservative sufficient conditions for fuzzy summations remains open. Many results for relaxed stabilization conditions have been published [6–8].

For design problems involving other performance requirements, such as response speed, constraints on control input and output and robust stabilization, some interesting results for T–S fuzzy control have been published. Kruszewski et al.

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considered discrete-time uncertain nonlinear models in a T–S form and studied their stabilization using a nonquadratic Lyapunov function [9]. Qiu et al. investigated delay-dependent robust $\mathcal{H}_\infty$ filtering for a class of uncertain discrete-time state-delayed T–S fuzzy systems [10]. Most $\mathcal{H}_\infty$ control formulations based on T–S fuzzy models concentrate on robust stability conditions and robust performance analysis, but without taking time-domain constraints into account.

Model predictive control (MPC), or receding horizon control (RHC), is one of the most powerful optimizing control approaches for a large class of constrained nonlinear industrial processes in which a control sequence is determined by optimizing a finite horizon cost at each sampling instant, based on an explicit process model and state measurement [11]. Many successful applications of MPC using fuzzy models have been reported. An efficient offset-free predictive control approach with output feedback for nonlinear processes based on their approximate fuzzy models was developed by Zhang et al. [12]. Considering the output and input constraints, Khairy et al. used a fuzzy model to predict future behavior and designed a stabilizing model-based predictive controller [13].

For robust MPC, Zhang et al. proposed two stable fuzzy-model predictive controllers based on piecewise Lyapunov functions and the min–max optimization of a quasi-worst-case infinite-horizon objective function [14]. However, for strong constraints and/or uncertainty, the monotonicity required for the value function needs further discussion.

Building on the work of Chen and Scherer [15], moving-horizon constrained $\mathcal{H}_\infty$ control based on T–S fuzzy models for a constrained discrete-time nonlinear system is presented. In the nominal case, MPC and T–S fuzzy models have been combined to achieve exponential stability of nonlinear systems subject to output and input constraints [13]. For robust MPC, this paper combines moving-horizon constrained $\mathcal{H}_\infty$ control and T–S fuzzy models to address the disturbance attenuation issue. First, the nonlinear system considered is represented by local models according to fuzzy rules, and the local models are blended into an overall single model through fuzzy membership functions. For this T–S fuzzy model, following Chen and Scherer [15], a parameter-dependent state feedback control law is adopted and the corresponding optimization problem is reduced to a convex optimization problem involving linear matrix inequalities (LMIs). This leads to a tractable nonlinear fuzzy moving-horizon $\mathcal{H}_\infty$ control problem for which the $\mathcal{H}_\infty$ attenuation index is adapted online to satisfy time-domain constraints.

The remainder of the paper is organized as follows. Section 2 introduces the constrained nonlinear system to be controlled and the T–S fuzzy models. Section 3 provides some preparatory results on LMI-based fuzzy constrained $\mathcal{H}_\infty$ control problems. An efficient algorithm for moving-horizon nonlinear $\mathcal{H}_\infty$ control is introduced in Section 4. The corresponding synthesis problem can be formulated as an LMI problem. In Section 5, the proposed controller is applied to a continuous stirred tank reactor and active queue management at routers. The simulation results demonstrate the effectiveness of the proposed algorithm. Section 6 concludes with a brief summary.

2. Preliminaries

2.1. System representation

Consider the smooth nonlinear control system

$$\begin{cases} x(k+1) = f(x(k), u(k), v(k)), & x(k) = x_k, \ k \geq 0, \\ z_1(k) = g_1(x(k), u(k), v(k)), \\ z_2(k) = g_2(x(k), u(k)) \end{cases} \tag{1}$$

subject to time-domain constraints

$$-z_{2s,\max} \leq z_2(k) \leq z_{2s,\max} \quad \text{for all } k \geq 0, \ s = 1, 2, \ldots, p, \tag{2}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are the state and control inputs, respectively, $v(k) \in \mathbb{R}^q$ is the disturbance, which belongs to $l_2[0, \infty)$, and $z_1(k) \in \mathbb{Z}_1 \subset \mathbb{R}^{p_1}$ and $z_2(k) \in \mathbb{Z}_2 \subset \mathbb{R}^{p_2}$ are the performance output and the output to be constrained, respectively. The origin $(0, 0)$ belongs to the set $\mathbb{Z}_2$, and $z_{2,\max}$ is the given constant-compatible vector. The nonlinear functions $f(\cdot, \cdot, \cdot)$, $g_1(\cdot, \cdot, \cdot)$ and $g_2(\cdot, \cdot)$ are twice continuously differentiable in all arguments. For simplicity of notation, we assume that the origin is an equilibrium in (1) without disturbances, i.e. $f(0, 0, 0) = 0$, $g(0, 0) = 0$.

The task is to design a controller for (1) such that the closed-loop system is asymptotically stable and the $\mathcal{H}_\infty$ norm from disturbance v to performance output z_1 is minimized while the time-domain constraints (2) are satisfied.

2.2. T–S fuzzy models and parallel distributed compensation

It has been proven that any twice differentiable continuous nonlinear function can be approximated, to any degree of accuracy, using T–S fuzzy models [2]. The main feature of the T–S fuzzy model is to express the local dynamics of each fuzzy rule by a linear system model. The overall fuzzy model of the system is obtained by fuzzy blending of the linear system models.

The ith rule of the T–S fuzzy model of the nonlinear system (1) is of the following form:

$\mathbf{R}^i$:

IF $\quad\lambda_1(k)$ is M_1^i, $\lambda_2(k)$ is M_2^i, …, and $\lambda_{r_1}(k)$ is $M_{r_1}^i$,

THEN $\quad x(k+1) = A_i x(k) + B_{1i} v(k) + B_{2i} u(k),$

$\qquad z_1(k) = C_{1i} x(k) + D_{11i} v(k) + D_{12i} u(k),$

$\qquad z_2(k) = C_{2i} x(k) + D_{22i} u(k),$

$\qquad i = 1, 2, \ldots, r,$

where $\mathbf{R}^i$ represents the ith fuzzy rule, M_j^i is the fuzzy set and r is the number of model rules. For $\lambda(k) = [\lambda_1(k),$ $\lambda_2(k), \ldots, \lambda_{r_1}(k)] \in \mathbb{R}^{r_1}$, premise variables $\lambda_j(k)$ may be functions of the state variables, external inputs, and/or time.

Nonlinear system (1) can be approximated in the form of a T–S fuzzy model as

$$\begin{cases} x(k+1) = \sum_{i=1}^{r} h_i(\lambda)[A_i x(k) + B_{1i} v(t) + B_{2i} u(k)], \\ z_1(k) = \sum_{i=1}^{r} h_i(\lambda)[C_{1i} x(k) + D_{11i} v(k) + D_{12i} u(k)], \\ z_2(k) = \sum_{i=1}^{r} h_i(\lambda)[C_{2i} x(k) + D_{22i} u(k)], \end{cases} \tag{3}$$

where

$$h_i(\lambda(k)) = \frac{\mu_i(\lambda(k))}{\sum_{i=1}^{r} \mu_i(\lambda(k))}, \quad \mu_i(\lambda(k)) = \prod_{j=1}^{r_1} M_j^i(\lambda_j(k)).$$

The term $M_j^i(\lambda_j(k))$ is the grade of membership for $\lambda_j(k)$ in M_j^i.

Since

$$\mu_i(\lambda(k)) \geq 0, \quad \sum_{i=1}^{r} \mu_i(\lambda(k)) > 0, \ i = 1, 2, \ldots, r,$$

we have

$$h_i(\lambda(k)) \geq 0, \quad \sum_{i=1}^{r} h_i(\lambda(k)) = 1, \ i = 1, 2, \ldots, r.$$

In the parallel distributed compensation (PDC) design [2,16], each control rule is designed from the corresponding rule of a T–S fuzzy model. The fuzzy controller designed shares the same fuzzy sets as the fuzzy model in the premise parts. Hence, we construct the following fuzzy controller via PDC:

$\mathbf{C}^i$:

IF $\quad\lambda_1(k)$ is M_1^i, $\lambda_2(k)$ is M_2^i, …, and $\lambda_{r_1}(k)$ is $M_{r_1}^i$, $\qquad\qquad\qquad\qquad$ (4)

THEN $\quad u(k) = K_i x(k), \quad i = 1, 2, \ldots, r,$

where $K_i \in \mathbb{R}^{m \times n}$ is a constant feedback matrix. The whole controller can be expressed as

$$u(k) = \kappa(\lambda(k))x(k), \tag{5}$$

where $\kappa(\lambda(k)) = \sum_{i=1}^r h_i(\lambda(k))K_i$. Substituting (5) into (3), we obtain the closed-loop system model

$$\begin{cases} x(k+1) = A_{cl}(\lambda(k))x(k) + B_{cl}(\lambda(k))v(k), \\ z_1(k) = C_{1cl}(\lambda(k))x(k) + D_{1cl}(\lambda(k))v(k), \\ z_2(k) = C_{2cl}(\lambda(k))x(k), \end{cases} \tag{6}$$

where

$$A_{cl}(\lambda(k)) = \sum_{i=1}^r \sum_{j=1}^r h_i(\lambda(k))h_j(\lambda(k))(A_i + B_{2i}K_j), \tag{7a}$$

$$C_{1cl}(\lambda(k)) = \sum_{i=1}^r \sum_{j=1}^r h_i(\lambda(k))h_j(\lambda(k))(C_{1i} + D_{12i}K_j), \tag{7b}$$

$$C_{2cl}(\lambda(k)) = \sum_{i=1}^r \sum_{j=1}^r h_i(\lambda(k))h_j(\lambda(k))(C_{2i} + D_{22i}K_j), \tag{7c}$$

$$B_{cl}(\lambda(k)) = \sum_{i=1}^r h_i(\lambda(k))B_{1i}, \tag{7d}$$

$$D_{1cl}(\lambda(k)) = \sum_{i=1}^r h_i(\lambda(k))D_{12i}. \tag{7e}$$

Our control problem has been converted to finding a parameter-dependent control law $\kappa(\lambda(k))$ such that closed-loop system (6) is asymptotically stable, the $\mathcal{H}_\infty$ norm from disturbance v to performance output z_1 is minimized, and the constrained output z_2 of the system meets the time-domain constraints.

Synthesis and analysis of a fuzzy control system can usually be reduced sufficiently to a feasibility problem with parameter-dependent matrix inequalities [2]. Here, we introduce a less conservative scheme to convert the parameter-dependent matrix inequalities to LMIs.

Lemma 1 (*Gao [17]*). *If there exist matrices $\mathcal{T}_{ii} = \mathcal{T}_{ii}^T$, $\mathcal{T}_{ij} = \mathcal{T}_{ji}^T$ ($i \neq j = 1, 2, \ldots, r$) such that $\mathcal{L}_{ij}$ ($1 \leq i, j \leq r$) satisfies*

$$\mathcal{L}_{ii} \leq \mathcal{T}_{ii}, \quad i = 1, 2, \ldots, r, \tag{8a}$$

$$\mathcal{L}_{ij} + \mathcal{L}_{ji} \leq \mathcal{T}_{ij} + \mathcal{T}_{ij}^T, \quad j < i, \tag{8b}$$

$$[\mathcal{T}_{ij}]_{r \times r} \leq 0, \tag{8c}$$

then the parameter-dependent matrix inequalities

$$\sum_{i=1}^r \sum_{j=1}^r h_i(\lambda(k))h_j(\lambda(k))\mathcal{L}_{ij} \leq 0 \tag{9}$$

are feasible, where $h_i(\lambda(k)) \geq 0$, $\sum_{i=1}^r h_i(\lambda(k)) = 1$, $\forall \lambda(k)$ and

$$[\mathcal{T}_{ij}]_{r \times r} = \begin{pmatrix} \mathcal{T}_{11} & \cdots & \mathcal{T}_{1r} \\ \vdots & \ddots & \vdots \\ \mathcal{T}_{r1} & \cdots & \mathcal{T}_{rr} \end{pmatrix}.$$

Remark 2.1. For the case "< 0", we obtain similar results.

3. Constrained $\mathcal{H}_\infty$ control based on T–S models

For the constrained $\mathcal{H}_\infty$ control problem, we can state the following result for closed-loop system (6).

Theorem 1 (*Constrained nonlinear $\mathcal{H}_\infty$ control based on T–S models*). *For a given α, suppose that the optimization problem*

$$\min_{\gamma, Q, Y_1, Y_2, \ldots, Y_r} \gamma \tag{10}$$

subject to

$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\lambda) h_j(\lambda) \mathcal{L}_{ij} \geq 0, \tag{11}$$

$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\lambda) h_j(\lambda) \mathcal{F}_{ij,s} \geq 0, \quad s = 1, 2, \ldots, n_{z2} \tag{12}$$

has an optimal solution (γ, Q, Y_1, Y_2, $\ldots$, Y_r). *In (11) and (12), $\mathcal{L}_{ij}$ and $\mathcal{F}_{ij,s}$ are given as*

$$\mathcal{L}_{ij} = \begin{bmatrix} Q & * & * & * \\ 0 & \gamma I & * & * \\ A_i Q + B_{2i} Y_j & B_{1i} & Q & 0 \\ C_{1i} Q + D_{12i} Y_j & D_{11i} & 0 & \gamma I \end{bmatrix},$$

$$\mathcal{F}_{ij,s} = \begin{bmatrix} \frac{1}{\alpha} z_{2s,\max}^2 & e_s^T (C_{2i} Q + D_{22i} Y_j) \\ * & Q \end{bmatrix},$$

where e_s is the element of the basis vector in $\mathbb{R}^{p_2}$ and $*$ denotes expressions that can be deduced by symmetry. Then (1) controlled with the parameter-varying state feedback control $u = \kappa(\lambda)x$ has the following properties.

 (I) *The $\mathcal{H}_\infty$ norm from disturbance v to performance output z_1 is less than (or equal to) γ.*
(II) *If the disturbance energy $\sum_{i=0}^{\infty} \|v(i)\|^2 \leq \beta$ and the initial state $x(0)$ satisfy $\gamma\beta + V(x(0)) \leq \alpha$, then*
     (i) *All perturbed state trajectories remain in an ellipsoid defined as*

$$\Omega(P, \alpha) := \{x \in \mathbb{R}^n | V(x) \leq \alpha\}. \tag{13}$$

     (ii) *The constraints (2) are respected.*

In the above, $\kappa(\lambda) = \sum_{j=1}^{r} h_j(\lambda) K_j$ with $K_j = Y_j Q^{-1}$, $V(x) = x^T P x$ with $P = Q^{-1}$.

Proof. (I) Consider the Lyapunov function candidate $V(x) = x^T P x$ with $P = P^T > 0$. Then the $\mathcal{H}_\infty$ performance of closed-loop system (6) is bounded by γ if the following dissipation inequality is satisfied [18]:

$$V(x(k+1)) - V(x(k)) + \gamma^{-1} z_1(k)^T z_1(k) - \gamma v(k)^T v(k) \leq 0. \tag{14}$$

By taking the parameter-varying state-feedback control (5) and substituting (6) into (14), we have

$$\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}^T \begin{bmatrix} A_{cl}^T(\cdot) P A_{cl}(\cdot) - P + \gamma^{-1} C_{1cl}^T(\cdot) C_{1cl}(\cdot) & A_{cl}^T(\cdot) P B_{cl}(\cdot) + \gamma^{-1} C_{1cl}^T(\cdot) D_{1cl}(\cdot) \\ B_{cl}^T(\cdot) P A_{cl}(\cdot) + \gamma^{-1} D_{1cl}^T(\cdot) C_{1cl}(\cdot) & B_{cl}^T(\cdot) P B_{cl}(\cdot) + \gamma^{-1} D_{1cl}^T(\cdot) D_{1cl}(\cdot) - \gamma \end{bmatrix}$$

$$\begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \leq 0, \tag{15}$$

where $A_{cl}(\cdot) = A_{cl}(\lambda(k))$, $B_{cl}(\cdot) = B_{cl}(\lambda(k))$, $C_{1cl}(\cdot) = C_{1cl}(\lambda(k))$ and $D_{1cl}(\cdot) = D_{1cl}(\lambda(k))$. Then (15) reduces to the condition that P satisfies

$$\begin{bmatrix} A_{cl}^T(\cdot)PA_{cl}(\cdot) - P + \gamma^{-1}C_{1cl}^T(\cdot)C_{1cl}(\cdot) & A_{cl}^T(\cdot)PB_{cl}(\cdot) + \gamma^{-1}C_{1cl}^T(\cdot)D_{1cl}(\cdot) \\ B_{cl}^T(\cdot)PA_{cl}(\cdot) + \gamma^{-1}D_{1cl}^T(\cdot)C_{1cl}(\cdot) & B_{cl}^T(\cdot)PB_{cl}(\cdot) + \gamma^{-1}D_{1cl}^T(\cdot)D_{1cl}(\cdot) - \gamma \end{bmatrix} \leq 0 \quad (16)$$

and the inequality (16) is equivalent to

$$\begin{bmatrix} P & * & * & * \\ 0 & \gamma I & * & * \\ PA_{cl}(\lambda(k)) & PB_{cl}(\lambda(k)) & P & 0 \\ C_{1cl}(\lambda(k)) & D_{1cl}(\lambda(k)) & 0 & \gamma I \end{bmatrix} \geq 0 \quad (17)$$

according to the classical Schur complement. By substituting (7) into (17), we infer that (14) can be guaranteed by the existence of $P > 0$ and K_j ($j = 1, 2, \ldots, r$) satisfying

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\lambda)h_j(\lambda) \begin{bmatrix} P & * & * & * \\ 0 & \gamma I & * & * \\ P(A_i + B_{2i}K_j) & PB_{1i} & P & 0 \\ C_{1i} + D_{12i}K_j & D_{12i} & 0 & \gamma I \end{bmatrix} \geq 0. \quad (18)$$

With substitution of $Q = P^{-1}$ and $Y_j = K_j Q$, we observe that the equivalence between (18) and (11) is obtained by performing a congruence transformation with $diag\{Q, I, Q, I\}$.

(II-i) Assuming $\sum_{i=0}^{\infty} \|v(i)\|^2 \, dt \leq \beta$, (14) implies that

$$V(x(k)) + \gamma^{-1}\sum_{i=0}^{k-1} \|z_1(k)\|^2 \leq V(x(0)) + \gamma\beta \quad (19)$$

for all $k \geq 0$. Given any $x(0)$, (19) shows that the state trajectory starting from $x(0)$ stays in the ellipsoid defined by (13) with $\alpha := \gamma\beta + V(x(0))$. This means that the ellipsoid $\Omega(P, \alpha)$ contains the set of all reachable states for the closed-loop system.

(II-ii) Following [19], we infer for all $k \geq 0$ that

$$|z_{2s}(k)|^2 = |e_s^T C_{2cl}(\lambda(k))x(k)|^2$$

$$= \left| e_s^T \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\lambda(k))h_j(\lambda(k))(C_{2i} + D_{22i}K_j)x(k) \right|^2$$

$$\leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\lambda(k))h_j(\lambda(k))|e_s^T(C_{2i} + D_{22i}K_j)x(k)|^2$$

(by virtue of $x(k) \in \Omega(P, \alpha)$)

$$\leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\lambda(k))h_j(\lambda(k)) \max_{x \in \Omega(P,\alpha)} |e_s^T(C_{2i} + D_{22i}Y_iQ^{-1})x|^2$$

(using the Cauchy–Schwarz inequality)

$$\leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\lambda(k))h_j(\lambda(k))\alpha\|e_s^T(C_{2i} + D_{22i}Y_i)Q^{-1/2}\|_2^2.$$

Using (12), we arrive at

$$|z_{2s}(k)| \leq z_{2s,\max}. \quad (20)$$

Hence, the feasibility of (12) implies that the time-domain constraints in (2) are respected. $\square$

From (12), we know that the larger the value of α, the smaller is the feasible set of $(Q, Y_1, Y_2, \ldots, Y_r)$ and hence the larger is the optimal value γ. This implies worse performance. However, property II(ii) reveals that the smaller the value of α, the smaller is the disturbance energy that guarantees satisfaction of the time-domain constraints. This contradiction motivates us to exploit the moving-horizon strategy to online manage the trade-off between constraint satisfaction and good performance.

4. Moving-horizon nonlinear $\mathcal{H}_\infty$ control based on a T–S model

At time k, nonlinear moving-horizon $\mathcal{H}_\infty$ control [15] based on a T–S model can be formulated as

$$\min_{\gamma, Q, Y_1, Y_2, \ldots, Y_r} \gamma \quad \text{s.t. (11), (12), and} \tag{21a}$$

$$\begin{bmatrix} \alpha - \gamma\beta & x(k)^T \\ x(k) & Q \end{bmatrix} \geq 0, \tag{21b}$$

$$\begin{bmatrix} p_0 - p_{k-1} + x^T(k)P_{k-1}x(k) & x(k)^T \\ x(k) & Q \end{bmatrix} \geq 0, \tag{21c}$$

where $(\gamma, Q, Y_1, Y_2, \ldots, Y_r)$ are the independent variables. That is, the values of $x(k)$, α and β are given for the solution at time $k = 0$ (constraint (21c) drops), and the values of $x(k)$, α, β, p_0 and P_{k-1} are given for the solution at time $k > 0$. Then the independent variables $\gamma, Q, Y_1, \ldots, Y_r$ in (21) are to be determined at time k. Constraint (21b) forces the actual state $x(k)$ to be in the interior of the ellipsoid $\Omega(P, \alpha - \gamma\beta)$ and dissipation constraint (21c) is required to guarantee that the closed-loop moving-horizon system is dissipative. With the actual state $x(k)$, if (21) has an optimal solution, denoted as $(\gamma_k, Q_k, Y_{1k}, Y_{2k}, \ldots, Y_{rk})$, we then define a feedback control law according to the MPC principle as follows:

$$u(k) = \sum_{j=1}^{r} h_j(\lambda)K_{jk}x(k) \tag{22}$$

with $K_{jk} = Y_{jk}Q_k^{-1}$. The scalar p_k in (21c) is recursively updated as

$$p_k := p_{k-1} - (x^T(k)P_{k-1}x(k) - x^T(k)P_kx(k)) \tag{23}$$

with $P_k = Q_k^{-1}$. A more detailed discussion on the dissipation constraint can be found in the literature [15,20].

To solve the parameter-dependent optimization problem (21) efficiently online, we reduce inequality constraints (11) and (12) to LMIs according to Lemma 1. That is, (11) and (12) are feasible if there exist $\mathcal{T}_{ij}$ and $\mathcal{M}_{ij}$ $(i, j = 1, 2, \ldots, r)$ such that

$$\begin{cases} \mathcal{L}_{ii} \geq \mathcal{T}_{ii}, \quad i = 1, 2, \ldots, r, \\ \mathcal{L}_{ij} + \mathcal{L}_{ji} \geq \mathcal{T}_{ij} + \mathcal{T}_{ij}^T, \quad j < i, \\ [\mathcal{T}_{ij}]_{r \times r} \geq 0. \end{cases} \tag{24a}$$

$$\begin{cases} \mathcal{F}_{ii,s} \geq \mathcal{M}_{ii,s}, \quad i = 1, 2, \ldots, r, \\ \mathcal{F}_{ij,s} + \mathcal{F}_{ji,s} \geq \mathcal{M}_{ij,s} + \mathcal{M}_{ij,s}^T, \quad j < i, \\ [\mathcal{M}_{ij,s}]_{r \times r} \geq 0. \end{cases} \tag{24b}$$

Hence, optimization problem (21) becomes

$$\min_{\gamma, Q, Y_j} \gamma \quad \text{s.t. (24a), (24b), (21b), and (21c) hold.} \tag{25}$$

Here, the independent variables are the same as in (21). By virtue of the moving horizon principle of MPC, optimization problem (21) is solved at each sampling time k, updated with the actual state $x(k)$ [11]. This is implementable, since p_{k-1} and P_{k-1} have been determined at the previous sampling time $k - 1$ and are held fixed. The procedure for solving the moving-horizon $\mathcal{H}_\infty$ control problem online based on T–S models can be summarized as follows.

Algorithm 1.

- Step 0. Obtain the T–S fuzzy model for the controlled system and choose (determine) parameters α and β for LMI optimization problem (21b).
- Step 1. At time $k = 0$, get $x(0)$ and solve (25) without (21c) to obtain (γ_0, P_0, Y_{j0}). Compute $p_0 = x(0)^T P_0 x(0)$ and go to step 3.
- Step 2. At time $k > 0$, get $x(k)$ and solve (25) to obtain an optimal solution (γ_k, P_k, Y_{jk}).
- Step 3. Set $K_{jk} = Y_{jk} Q_k^{-1}$ and $p_k = p_{k-1} - (x^T(k) P_{k-1} x(k) - x^T(k) P_k x(k))$. Apply closed-loop control (22) to the plant, replace k by $k + 1$ and go to step 2.

Theorem 2 (*Moving-horizon $\mathcal{H}_\infty$ control based on T–S models*). *For given (α, β), suppose that semi-definite programming problem (25) with actual state $x(k)$ has an optimal solution (γ_k, Q_k, Y_{jk}) at any time k. Then the parameter-varying state feedback (22) guarantees the following.*

(I) *The dissipation inequality*

$$\sum_{i=0}^{k} (\bar{\gamma}^{-1} \parallel z_1(i) \parallel^2 - \bar{\gamma} \parallel v(i) \parallel^2) \leq x(0)^T P_0 x(0) \tag{26}$$

is satisfied, with $\bar{\gamma} := \max\{\gamma_0, \gamma_1, \ldots, \gamma_k\}$.

(II) *The $\mathcal{H}_\infty$ norm from disturbance v to performance output z_1 is less than $\bar{\gamma}$.*

(III) *The constraints (2) are satisfied.*

Proof. (I) Because of the Schur complement, (21c) is equivalent to

$$p_0 - p_{k-1} + x(k)^T P_{k-1} x(k) - x(k)^T P_k x(k) \geq 0. \tag{27}$$

Substituting (23) into the above matrix inequality recursively, we obtain that dissipation constraint (21c) enforces

$$\sum_{i=1}^{k} (x(i)^T P_{i-1} x(i) - x(i)^T P_i x(i)) \geq 0. \tag{28}$$

The feasibility of (24a) at sampling time k implies that inequality (14) is satisfied with $V(x) = x^T P_k x$ and γ_k, i.e.

$$x(k+1)^T P_k x(k+1) - x(k)^T P_k x(k) + \gamma_k^{-1} z(k)^T z(k) - \gamma_k w(k)^T w(k) < 0. \tag{29}$$

Hence, cumulating (29) from $i = 0$ to k leads to

$$\sum_{i=0}^{k} \bar{\gamma}^{-1} \|z_1(i)\|^2 - \bar{\gamma} \|w(i)\|^2 \leq x(0)^T P_0 x(0) - x(k+1)^T P_l x(k+1) - \sum_{i=1}^{k} (x(i)^T P_{i-1} x(i) - x(i)^T P_i x(i)), \tag{30}$$

where $\bar{\gamma} = \max\{\gamma_0, \gamma_1, \ldots, \gamma_k\}$. Considering (28), the above inequality becomes

$$\sum_{i=0}^{k} \bar{\gamma}^{-1} \|z_1(i)\|^2 - \bar{\gamma} \|v(i)\|^2 \leq x(0)^T P_0 x(0) - x(k+1)^T P_k x(k+1). \tag{31}$$

Thus, dissipation inequality (26) is satisfied, since $x(k+1)^T P_k x(k+1) \geq 0$.

(II) Since $x(0)^T P_0 x(0) \geq 0$, we furthermore obtain through (31) that the $\mathcal{H}_\infty$ norm from disturbance v to performance output z_1 is less than $\bar{\gamma}$.

(III) From the proof of Theorem 1 and Lemma 1, we know that constraints (2) are satisfied at any time k. Thus, the argument can be continued as time elapses. $\square$

Remark 4.1. In robust MPC, we strive in general to solve the following minimax optimization problem for system (3) and actual state $x(k)$ in the moving-horizon fashion:

$$\min_u \max_v \sum_{i=k}^{\infty} \|z_1(i)\|^2.$$

Similar to the approach of Chen et al. [21], the above minimax problem for the fuzzy control system can be relaxed to a Lagrange (weak) dual problem. By a similar derivation, the Lagrange (weak) dual problem is equivalent to inequality constraint (11) in optimization problem (21) with the feedback law $u = Kx$. Then, solving (21) gives a solution of the primary minimax problem. Moreover, it is solved repeatedly, with the actual state used to update (11). It should be pointed out that the proposed approach is performed implicitly an infinite horizon prediction with feedback parameterization of $u = Kx$, which is clearly different from classical MPC.

Remark 4.2. Compared to studies in which a static state-feedback law is applied during the sampling interval [15,22,23], we adopt a linear parameter variable control law [Eq. (22)], which introduces extra degrees of freedom in our optimization problem.

Remark 4.3. Dissipation constraint (21c) can automatically adapt the performance level while respecting time domain constraints. Moreover, because of recursion of (23) with state p_k, the level of dissipation does not have to decrease in each step. This makes it possible to reduce the infeasibility of (25). However, if the system is affected by large disturbances, (25) may be still infeasible. In this case, the controller has to maintain the previous values as remedial action. At the next sampling time, (25) is solved with the new states.

5. Illustrative examples

5.1. Nonlinear stirred tank reactor

In this section, the proposed robust MPC algorithm is applied to a benchmark system, a continuous stirred tank reactor (CSTR) [24,25]. Assuming a constant liquid volume, the CSTR for an exothermic, irreversible reaction A $\longrightarrow$ B is described by the following dynamic model based on a component balance for reactant A and an energy balance:

$$\begin{cases} \dot{C}_A = \dfrac{q}{V}(C_{Af} - C_A) - k_0 \exp\left(-\dfrac{E}{RT}\right) C_A, \\[2mm] \dot{T} = \dfrac{q}{V}(T_f - T) - \dfrac{\Delta H}{\rho C_p} k_0 \exp\left(-\dfrac{E}{RT}\right) C_A + \dfrac{UA}{V\rho C_p}(T_c - T) + v_T, \end{cases} \tag{32}$$

where C_A is the concentration of reactant A, T is the reactor temperature, and T_c is the temperature of the coolant steam. The parameters are $q = 100\,\text{l/min}$, $V = 100\,\text{l}$, $C_{Af} = 1\,\text{mol/l}$, $T_f = 350\,\text{K}$, $\rho = 10^3\,\text{g/l}$, $C_p = 0.239\,\text{J/(g K)}$, $k_0 = 7.2 \times 10^{10}\,\text{min}^{-1}$, $E/R = 8750\,\text{K}$, $\Delta H = -5 \times 10^4\,\text{J/mol}$, and $UA = 5 \times 10^4\,\text{J/(min K)}$. Under these conditions the steady state is $C_A^{eq} = 0.5\,\text{mol/l}$, $T_c^{eq} = 300\,\text{K}$, and $T^{eq} = 350\,\text{K}$, which is an unstable equilibrium. The temperature of the coolant steam is constrained to $250\,\text{K} \le T_c \le 350\,\text{K}$, the reactant concentration is constrained to $0.4\,\text{mol/l} \le C_A \le 0.6\,\text{mol/l}$, and the reactor temperature is constrained to $335\,\text{K} \le T \le 365\,\text{K}$.

Following Cao and Frank [26], the model is discretized using a sampling period of $T_s = 0.083\,\text{min}$ (5 s). The maximum uncertainty is bounded by $v_{\max}$, $v_{\max} = 10\,\text{K/min}$. The control objective is to minimize H_∞ performance from disturbance v_T to concentration C_A and reactor temperature T around the steady state, manipulating the temperature of the coolant in its admissible range for any possible disturbance.

In this example, we treat the term $\exp(-E/RT)$ as a parameter-varying term, where T is the only varying parameter. Accordingly, nonlinear system (32) can be rewritten as the linear parameter-varying system

$$\begin{cases} \dot{C}_A = \left[-\dfrac{q}{V} - k_0 \exp\left(-\dfrac{E}{RT}\right)\right] C_A + \dfrac{q}{V} C_{Af}, \\[2mm] \dot{T} = -\dfrac{\Delta H}{\rho C_p} k_0 \exp\left(-\dfrac{E}{RT}\right) C_A - \left[\dfrac{q}{V} + \dfrac{UA}{V\rho C_p}\right] T + \dfrac{UA}{V\rho C_p} T_c + v_T + \dfrac{q}{V} T_f, \end{cases} \tag{33}$$

where the varying term $\exp(-E/RT) \in [\exp(-E/335R), \exp(-E/365R)]$ since R and E are constant, and the function $\exp(-E/RT)$ monotonically increases in the set $T \in [335, 365]$.

We normalize system (33), that is, we define states $x_1 = (C_A - C_A^{eq})/C_A^{eq}$, $x_2 = (T - T^{eq})/T^{eq}$, control input $u = (T_c - T_c^{eq})/T_c^{eq}$ and external $v = v_T/v_{\max}$, and then the nonlinear system can be approximated by the following two-rule T–S fuzzy model:

R^1 : IF reactor temperature is low (i.e., $x_2(t)$ is about -1),

  THEN $x(t) = A_1x(t) + B_{11}v(t) + B_{21}u(t)$

  $z_1(k) = C_{11}x(k) + D_{111}v(k) + D_{121}u(k)$,

  $z_2(k) = C_{21}x(k) + D_{221}u(k)$,

R^2 : IF reactor temperature is high (i.e., $x_2(t)$ is about 1),

  THEN $\dot{x}(t) = A_2x(t) + B_{12}v(t) + B_{22}u(t)$

  $z_1(k) = C_{12}x(k) + D_{112}v(k) + D_{122}u(k)$,

  $z_2(k) = C_{22}x(k) + D_{222}u(k)$,

with

$$A_1 = \begin{pmatrix} 0.8957 & 0 \\ 0.0007 & 0.7736 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.7299 & 0 \\ 0.0005 & 0.7736 \end{pmatrix},$$

$$B_{11} = B_{21} = \begin{pmatrix} 0 \\ 0.1531 \end{pmatrix}, \quad B_{12} = B_{22} = \begin{pmatrix} 0 \\ 0.0732 \end{pmatrix}.$$

For CSTR control, we choose $[x_1 \ x_2]^T$ as performance outputs, and hence

$$C_{1i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_{11i} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_{12i} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad i = 1, 2.$$

Moreover, the constrained outputs are $[x_1 \ x_2 \ u]^T$ and hence

$$C_{2,i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_{22i} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad i = 1, 2.$$

The membership functions shown in Fig. 1 are given by

$$h_1(\lambda) = \left(\exp\left(\frac{-8750}{T}\right) - \exp\left(\frac{-8750}{335}\right) \right) \Big/ \left(\exp\left(\frac{-8750}{365}\right) - \exp\left(\frac{-8750}{335}\right) \right),$$

$$h_2(\lambda) = \left(\exp\left(\frac{-8750}{365}\right) - \exp\left(\frac{-8750}{T}\right) \right) \Big/ \left(\exp\left(\frac{-8750}{365}\right) - \exp\left(\frac{-8750}{335}\right) \right)$$

and $\lambda(k) = T$ is the premise variable.

We choose $\alpha = 1.5$ and $\beta = 0.2$ in Algorithm 1. Fig. 2 shows the system state responses under nonlinear moving-horizon $\mathcal{H}_\infty$ control based on a T–S model (solid lines) with the zero initial condition; the corresponding disturbance input is plotted in Fig. 1. The proposed solutions are compared with those from a linear state feedback controller termed constrained $\mathcal{H}_\infty$ control (dotted line) [27]. The control input u and the performance level γ are plotted in Fig. 3.

From the plot of the input variables it is clear that the control actions calculated using constrained $\mathcal{H}_\infty$ control violate the input constraint. Thus, the closed-loop system under constrained $\mathcal{H}_\infty$ control will be unstable. By contrast, it is clear that moving-horizon $\mathcal{H}_\infty$ control respects the time-domain constraints by online relaxation of the performance level γ.

The computation time for this example needs to be addressed. Table 1 summarizes the computer configuration parameters used for simulation and Fig. 4 shows the computation time for solving (25) versus sampling time. The results
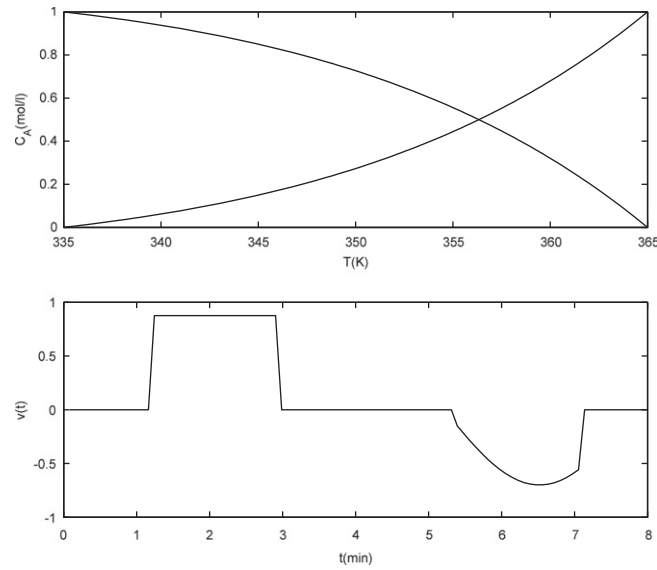
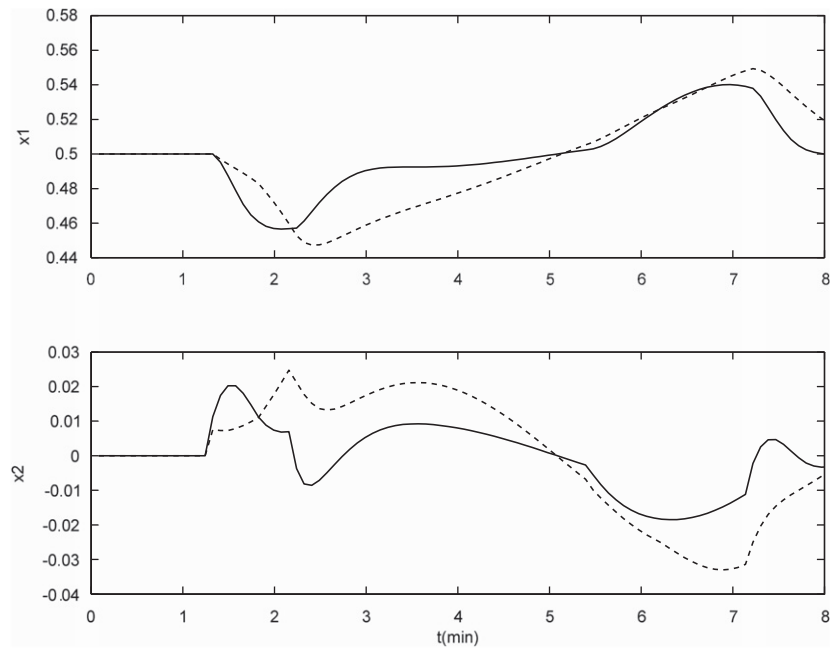Fig. 1. Membership functions of the fuzzy CSTR model and disturbance input $v(k)$.

Fig. 2. Responses of $x_1(k)$ and $x_2(k)$.

show that the computation time increases when disturbance exists in the system. Moreover, the average computation time is much less than the sampling period.

5.2. Active queue management

The proposed moving-horizon $\mathcal{H}_\infty$ approach was also used to design a controller for AQM routers [28]. Ignoring the TCP timeout mechanism, a dynamic model of TCP behavior is described by the following coupled and nonlinear
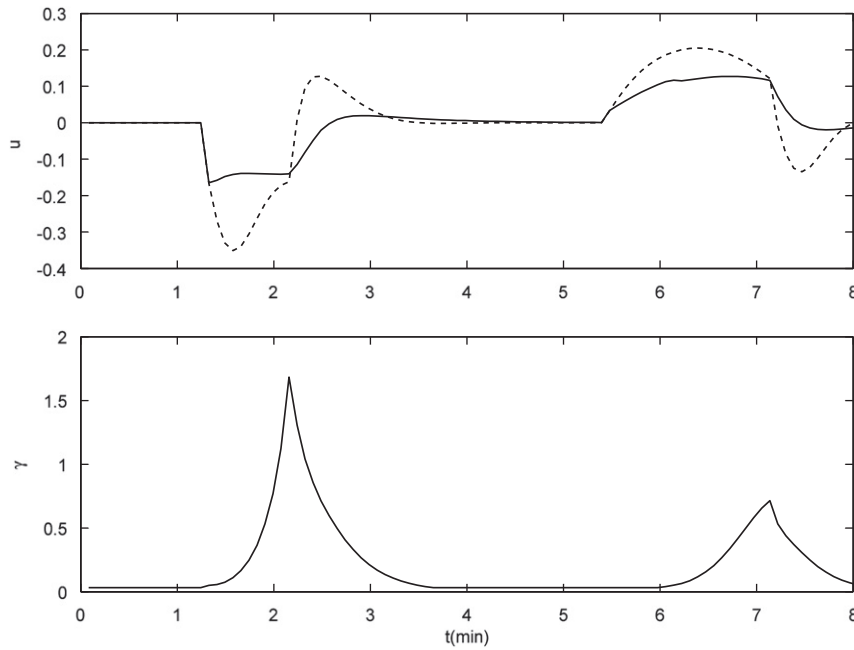
Fig. 3. Control input $u(k)$ and $\mathcal{H}_\infty$ performance index $\gamma(k)$.

Table 1
Computer configuration parameters.

Manufacturer and model	Intel Core 2 Duo
Processor speed	2.5 GHz
RAM size	2.00 GB
Hard drive size	160 GB

differential equations:

$$
\begin{cases}
\dot{w}(t) = \dfrac{w(t - R(t))}{w(t)R(t - R(t))}(1 - p(t - R(t))) - \dfrac{w(t)w(t - R(t))}{2R(t - R(t))}p(t - R(t)), \\[2mm]
\dot{q}(t) = -C(t) + N\dfrac{w(t)}{R(t)},
\end{cases}
\tag{34}
$$

where w denotes the average TCP window size (packets), q the average queue length (packets), C the link capacity (packets/s), $R(t) = q(t)/C(t) + T_p$ the transmission RTT (s), T_p the propagation delay (s), N the number of TCP sessions and p the drop probability of a packet.

The queue length, window size and drop probability are positive and bounded quantities, $q \in [0, q_{max}]$, $w \in [0, w_{max}]$, and $q \in [0, 1]$, where q_{max} and w_{max} denote the buffer capacity and maximum window size, respectively. Probability p belongs to the interval $[0, 1]$. In practical networks, the available link capacity changes with time and is difficult to measure. Here, we suppose that the nominal value, say C_0, of $C(t)$ is known, while $\delta C(t) = C(t) - C_0$ is unknown and considered as a disturbance to the system [29].

Take (w, q) as the state, p as the input and q as the output. For a given triplet of network parameters (N, C_0, T_p), any operating point (w_*, q_*, p_*) is defined by $\dot{w} = 0$ and $\dot{q} = 0$, so that $R_* = q_*/C_0 + T_p$, $w_* = R_*C_0/N$, $q_* = 2N^2/(2N^2 + C_0^2 R_*^2)$.
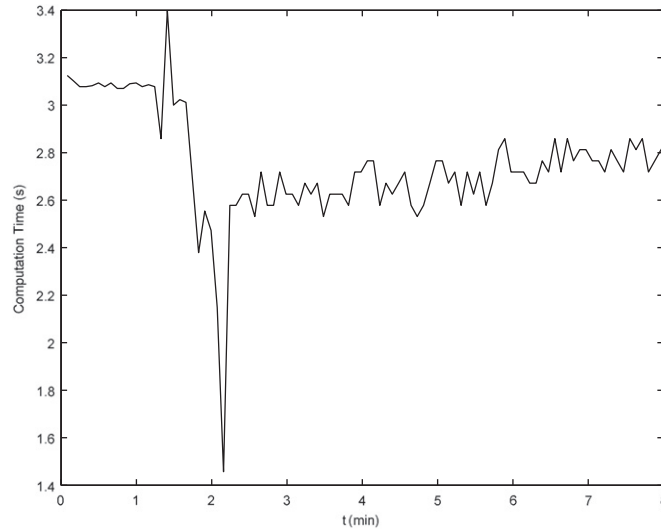
Fig. 4. Computation time for Algorithm 1 versus sampling time.

The equilibrium of model (34) is defined as (w_0, q_0, p_0) and then we set $\delta q = q - q_0$, $\delta p = p - p_0$, $\delta w = w - w_0$ and $\delta C = C - C_0$. System (34) has the form

$$\dot{\tilde{x}}(t) = F(\tilde{x}, \delta C, u) = f(\tilde{x}, \delta C) + g(\tilde{x}, \delta C)\tilde{u}(t - R_0), \qquad (35)$$

where

$$\tilde{x}(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} \delta w(t) \\ \delta q(t) \end{bmatrix}, \quad f(\tilde{x}, \delta C) = \begin{bmatrix} f_1(\tilde{x}, \delta C) \\ f_2(\tilde{x}, \delta C) \end{bmatrix}, \quad g(x, \delta C) = \begin{bmatrix} g_1(\tilde{x}, \delta C) \\ g_2(\tilde{x}, \delta C) \end{bmatrix},$$

$$f_1(\tilde{x}, \delta C) = \frac{(\tilde{x}_1(t - R_0) + w_0)(2 - p_0(2 + (\tilde{x}_1(t) + w_0)^2))}{2(\tilde{x}_1(t) + w_0)\left(\dfrac{\tilde{x}_2(t - R_0) + q_0}{\delta C(t - R_0) + C_0} + T_p \right)},$$

$$f_2(\tilde{x}, \delta C) = -(\delta C(t) + C_0) + n\frac{\tilde{x}_1(t) + w_0}{\left(\dfrac{\tilde{x}_2(t) + q_0}{\delta C(t) + C_0} + T_p \right)},$$

$$g_1(\tilde{x}, \delta C) = -\frac{(\tilde{x}_1(t - R_0) + w_0)(2 + (\tilde{x}_1 + w_0)^2)}{2(\tilde{x}_1 + w_0)\left(\dfrac{\tilde{x}_2(t - R_0) + q_0}{\delta C(t - R_0) + C_0} + T_p \right)},$$

$$g_2(\tilde{x}, \delta C) = 0, \quad \tilde{u}(t - R_0) = \delta p(t - R_0).$$

Obviously, the equilibrium point of system (35) is $(\tilde{x}_0 = 0, \tilde{u}_0 = 0)$. Ignoring the dependence of the time-delay argument $t - R(t)$ on queue length, the result of Taylor's linearization of model (35) about the equilibrium $(\tilde{x}_0, \tilde{u}_0)$ is obtained as [30]

$$\dot{\tilde{x}}(t) = \begin{bmatrix} \dfrac{-2C_0 n}{2n^2 + C_0^2 R_0^2} & 0 \\ \dfrac{n}{R_0} & -\dfrac{1}{R_0} \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} -\dfrac{2n^2 + R_0^2 C_0^2}{2n^2 R_0} \\ 0 \end{bmatrix} \tilde{u}(t - R_0) + \begin{bmatrix} 0 \\ -\dfrac{T_p}{R_0} \end{bmatrix} \delta C(t). \qquad (36)$$

About an operating non-equilibrium point $(\tilde{x}_s, \tilde{u}_s)$, the linearized model [31] has the form

$$\dot{\tilde{x}}(t) = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} -\dfrac{2n^2 + R_s^2 C_0^2}{2n^2 R_s} \\ 0 \end{bmatrix} \tilde{u}(t - R_0) + \begin{bmatrix} 0 \\ -\dfrac{T_p}{R_s} \end{bmatrix} \delta C(t), \qquad (37)$$

where

$$a_i = \nabla f_i(\tilde{x}_s, 0) + \frac{f_i(\tilde{x}_s, 0) - \tilde{x}_s^T \nabla f_i(\tilde{x}_s, 0)}{\|\tilde{x}_s\|^2} \tilde{x}_s, \quad \tilde{x}_s \neq 0, \ i = 1, 2. \tag{38}$$

Here, the change for the input delay is ignored. However, models (36) and (37) with an input time delay should be approximated by a common linear model. A simple method is to approximate the time delay with rational terms using an all-pole approximation, i.e., the transfer function of time delay $e^{-\tau s} \approx 1/(1 + \tau s)$. Then model (36) is approximated by

$$\begin{bmatrix} \delta \dot{w}(t) \\ \delta \ddot{w}(t) \\ \delta \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{a}{R_0} & \frac{aR_0 - 1}{R_0} & 0 \\ \frac{n}{R_0} & 0 & -\frac{1}{R_0} \end{bmatrix} \begin{bmatrix} \delta w(t) \\ \delta \dot{w}(t) \\ \delta q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{R_0} \\ 0 \end{bmatrix} \tilde{u}(t) + \begin{bmatrix} 0 \\ 0 \\ -\frac{T_p}{R_0} \end{bmatrix} \delta C(t), \tag{39}$$

where $a = -2C_0 n/(2n^2 + C_0^2 R_0^2)$ and $b = -(2n^2 + R_0^2 C_0^2)/2n^2 R_0$; model (37) can be approximated by the same way. According to the above linear local dynamic models, T–S fuzzy models can be immediately constructed. We choose q as the premise variable and use three fuzzy rules to construct the T–S fuzzy models.

Here, the parameters are $N = 120$, $C_0 = 3750$ packets/s, $T_p = 0.05$ s, $q_{max} = 600$ packets, $w_{max} = 20$. The maximum link bandwidth disturbance is approximately 6.4% of its nominal value. The control objective is to minimize H_∞ performance from the disturbance $\delta C(t)$ to the queue length $q(t)$ around the steady state $q_0 = 300$ packets and adapt the performance online to satisfy time-domain constraints.

Here, the equilibrium point of model (34) for the state transformation is

$$q_0 = 300, \quad w_0 = 4.0625, \quad p_0 = 0.1081, \quad R_0 = 0.13.$$

Then the other two operating points can be obtained as follows:

$$q^+ = 450, \quad w^+ = 5.3125, \quad p^+ = 0.0012, \quad R^+ = 0.17;$$

$$q^- = 150, \quad w^- = 2.8125, \quad p^- = 0.2018, \quad R^- = 0.09.$$

Three coordinate linear subsystems are obtained and reduced to dimensionless forms using the notation $x_1(t) = (w(t) - w_0)/(w_{max} - w_0)$, $x_2(t) = \dot{w}(t)/(w_{max} - w_0)$, $x_3(t) = (q(t) - q_0)/(q_{max} - q_0)$, control input $u(t) = (p(t) - p_0/(1 - p_0)$ and external disturbance $v(t) = (C(t) - C_0)/0.064C_0$. Finally, we discretize these using a sampling period $T_s = 0.15$ s. We define the membership functions for state variable $x_3(k)$ as in Fig. 5.

Then the nonlinear TCP network with AQM routers can be obtained by blending the following three linear subsystems.

$R^1 :$ IF the queue length $x_3(k)$ is M_{11}

THEN $x(k + 1) = A_1 x(k) + B_{11} v(k) + B_{21} u(k)$

$z_1(k) = C_{11} x(k) + D_{111} v(k) + D_{121} u(k)$,

$z_2(k) = C_{21} x(k) + D_{221} u(k)$, (40a)

$R^2 :$ IF the queue length $x_3(k)$ is M_{21}

THEN $x(k + 1) = A_2 x(k) + B_{12} v(k) + B_{22} u(k)$

$z_1(k) = C_{12} x(k) + D_{112} v(k) + D_{122} u(k)$,

$z_2(k) = C_{22} x(k) + D_{222} u(k)$, (40b)

$R^3 :$ IF the queue length $x_3(k)$ is M_{31}

THEN $x(k + 1) = A_3 x(k) + B_{13} v(k) + B_{23} u(k)$

$z_1(k) = C_{13} x(k) + D_{113} v(k) + D_{123} u(k)$,

$z_2(k) = C_{23} x(k) + D_{223} u(k)$, (40c)
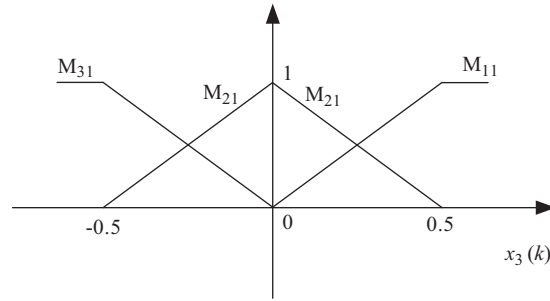
Fig. 5. Membership functions of $x_3(k)$.

with

$$A_1 = \begin{pmatrix} 0.8526 & 0.0925 & 0 \\ -1.1255 & 0.1170 & 0 \\ 4.0966 & 0.3033 & 0.3084 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.7392 & 0.0682 & 0 \\ -1.7715 & -0.0156 & 0 \\ 4.3123 & 0.2868 & 0.2147 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.5129 & 0.0364 & 0 \\ -2.5511 & -0.1212 & 0 \\ 3.9485 & 0.2172 & 0.1084 \end{pmatrix}, \quad B_{11} = \begin{pmatrix} 0 \\ 0 \\ -0.0346 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} -6.3360 \\ -48.3767 \\ -13.4356 \end{pmatrix},$$

$$B_{12} = \begin{pmatrix} 0 \\ 0 \\ -0.0393 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} -5.4957 \\ -43.7970 \\ -14.6205 \end{pmatrix}, \quad B_{13} = \begin{pmatrix} 0 \\ 0 \\ -0.0446 \end{pmatrix}, \quad B_{23} = \begin{pmatrix} -4.2526 \\ -22.2706 \\ -15.1520 \end{pmatrix}.$$

For design of a controller for AQM routers, we choose x_3 as the performance output and hence

$$C_{1i} = (0\ 0\ 1), \quad D_{11i} = 0, \quad D_{12i} = 0, \ i = 1, 2, 3.$$

Moreover, the constrained outputs are $[x_1\ x_3\ u]^T$ and hence

$$C_{2,i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_{22i} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ i = 1, 2, 3.$$

For T–S fuzzy model (40), the T–S fuzzy controller can be designed by applying Theorem 2. Thus, the T–S fuzzy controller has the form:

C^1 :   IF the queue length $x_3(k)$ is M_{11}

     THEN $u(k) = K_1 x(k)$,   (41a)

C^2 :   IF the queue length $x_3(k)$ is M_{21}

     THEN $u(k) = K_2 x(k)$,   (41b)

C^3 :   IF the queue length $x_3(k)$ is M_{31}

     THEN $u(k) = K_3 x(k)$.   (41c)

The corresponding closed-loop system maintains consistence with (6).

We choose $\alpha = 5.5$ and $\beta = 1$ in Algorithm 1. Figs. 6–8 show the system state responses and control input under the proposed moving-horizon $\mathcal{H}_\infty$ control (solid line) for the initial condition $x(0) = [0.308\ 0\ 0.5]^T$; the corresponding disturbance input is plotted in Fig. 9. The proposed solutions are compared with those given by the controller proposed by Chen and Scherer [15] (dotted line).
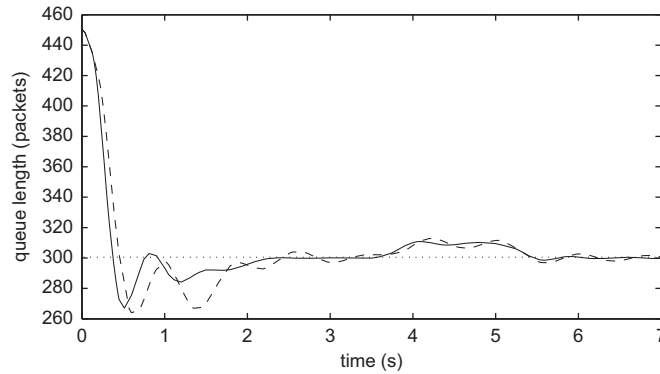
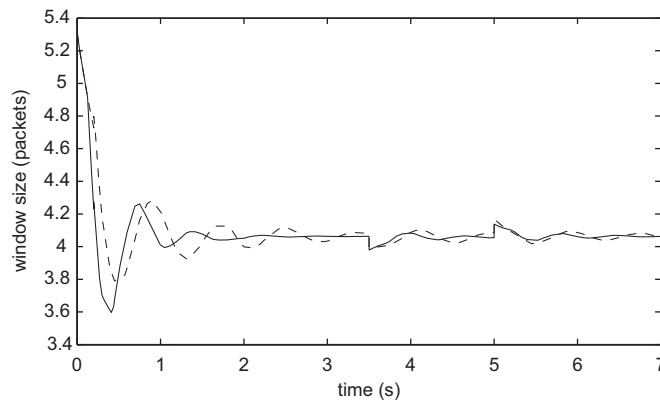Fig. 6. Response of the queue length.

Fig. 7. Response of the TCP window size.

The states for the proposed controller converge to the equilibrium point more rapidly if the states are far away from the equilibrium point and are disturbed. When the system states are close to the equilibrium point, the performance of the two control approaches is similar, although the Chen and Scherer control (dotted line) causes slight oscillation. The average drop probability of 10.96% calculated for moving-horizon $\mathcal{H}_\infty$ control is less than the 11.83% calculated for the Chen and Scherer approach. The same computer parameters as for the CSTR case were used. The computation time versus sampling time is plotted in Fig. 10. It is clear to see that the influence of disturbance on the computation time still exists. Compared to the CSTR example, the average computation time for AQM control is greater, which can be attributed to the number of rules for the controller. Moreover, the average computation time is greater than the sampling time.

Remark 5.1. The above two examples indicate that the computational effort for (25) with LMI constraints is strongly related to the number of controller rules. The computation time for Algorithm 1 is longer than the sampling time in the second example. Thus, it is clear that there is still much work to be done for systems with fast dynamics. One possibility for speeding up the online calculation is to implement the proposed controller on a field-programmable gate array (FPGA) chip [32]. A second possibility for fast computation is to develop a fast optimization algorithm [33].

6. Conclusions

For nonlinear systems with time-domain constraints, we proposed a moving-horizon $\mathcal{H}_\infty$ control algorithm based on T–S fuzzy models. The nonlinear dynamics are represented by local models according to fuzzy rule, and the local
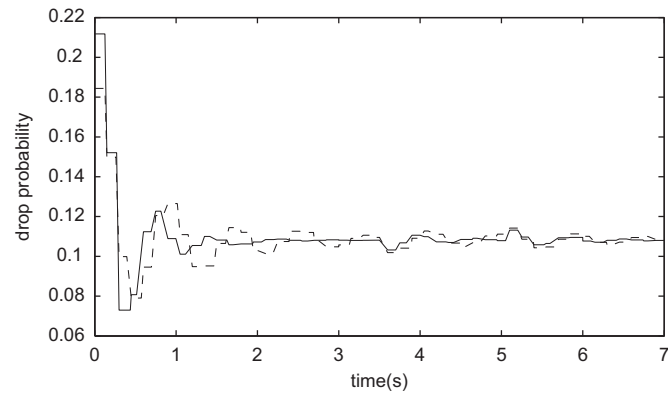
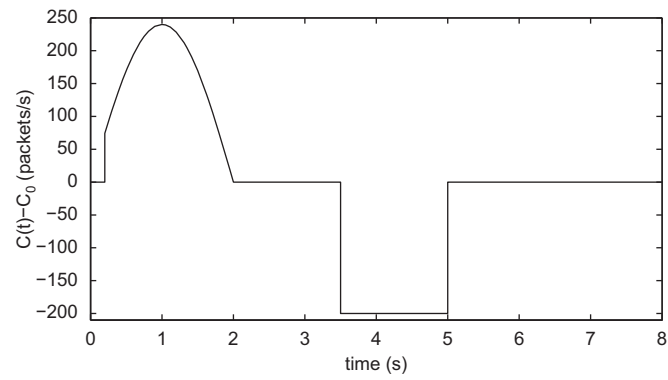Fig. 8. The drop probability used as control input.
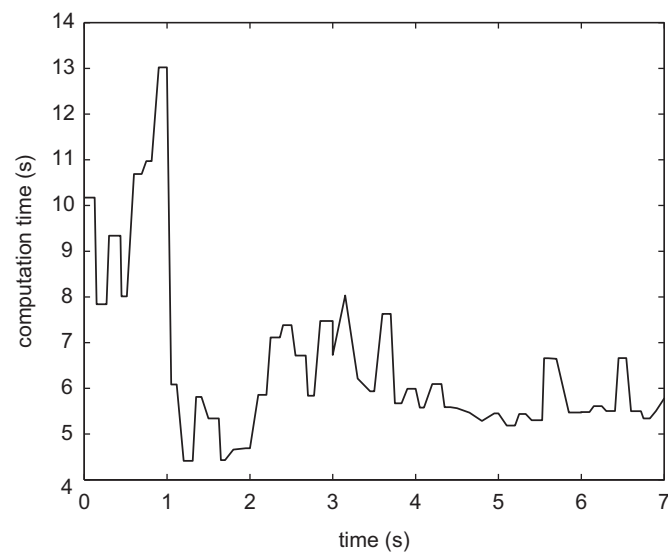

Fig. 9. The disturbance profile.

Fig. 10. The computation time for Algorithm 1 versus sampling time.

models are blended into an overall single model through fuzzy membership functions. A convex optimization problem involving LMIs is repeatedly solved online to guarantee disturbance attenuation. Similar to the moving-horizon $\mathcal{H}_\infty$ control for a constrained linear system [15], the proposed algorithm can trade off system performance against constraint satisfaction in an adaptive manner. Two application examples were presented to show the merits and design procedures of the proposed controller.

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References

[1] T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, IEEE Trans. Syst. Man Cybern. 15 (1985) 116–132.

[2] K. Tanaka, H.O. Wang, Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach, 1st ed., John Wiley & Sons, New York, 2002.

[3] D. Lee, J. Park, Y. Joo, A new fuzzy Lyapunov function for relaxed, stability condition of continuous-time Takagi–Sugeno fuzzy systems, IEEE Trans. Fuzzy Syst. 19 (2011) 785–791.

[4] B. Rhee, S. Won, A new fuzzy Lyapunov function approach for a Takagi–Sugeno fuzzy control system design, Fuzzy Sets Syst. 157 (2006) 1211–1228.

[5] M. Bernal, T.M. Guerra, Generalized non-quadratic stability of continuous-time Takagi–Sugeno models, IEEE Trans. Fuzzy Syst. 18 (2010) 815–822.

[6] C. Fang, Y. Liu, S.W. Kau, A new LMI-based approach to relaxed quadratic stabilization of T–S fuzzy control systems, IEEE Trans. Fuzzy Syst. 14 (2006) 386–397.

[7] A. Kruszewski, T. Sala, T. Guerra, C. Ariño, A triangulation approach to asymptotically exact conditions for fuzzy summations, IEEE Trans. Fuzzy Syst. 17 (2009) 985–994.

[8] A. Sala, C. Ariño, Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: applications of Polya's theorem, Fuzzy Sets Syst. 158 (2007) 2671–2686.

[9] A. Kruszewski, R. Wang, T. Guerra, Non-quadratic stabilization conditions for a class of uncertain non linear discrete-time T–S fuzzy models: a new approach, IEEE Trans. Autom. Control 53 (2008) 606–611.

[10] J.B. Qiu, G. Feng, J. Yang, A new design of delay-dependent robust H_∞ filtering for discrete-time T–S fuzzy systems with time-varying delay, IEEE Trans. Fuzzy Syst. 17 (2009) 1044–1058.

[11] D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, Constrained model predictive control: stability and optimality, Automatica 36 (2000) 789–814.

[12] T.J. Zhang, G. Feng, X.J. Zeng, Output tracking of constrained nonlinear processes with offset-free input-to-state stable fuzzy predictive control, Automatica 45 (2009) 900–909.

[13] M. Khairy, A.L. Elshafei, H.M. Emara, LMI based design of constrained fuzzy predictive control, Fuzzy Sets Syst. 161 (2010) 893–918.

[14] T.J. Zhang, G. Feng, J.H. Lu, Fuzzy constrained min–max model predictive control based on piecewise Lyapunov functions, IEEE Trans. Fuzzy Syst. 15 (2007) 686–698.

[15] H. Chen, C.W. Scherer, Moving horizon H_∞ control with performance adaptation for constrained linear systems, Automatica 42 (2006) 1033–1040.

[16] H.O. Wang, K. Tanaka, M. Grifin, Parallel distributed compensation of nonlinear systems by Tanaka–Sugeno fuzzy model, in: Proceedings of FUZZ-IEEE/IFES'95, 1995, pp. 531–538.

[17] X. Gao, Control for T–S Fuzzy Systems Based on LMI Optimization, Dissertation, Jilin University, 2006 (in Chinese).

[18] W. Lin, C.I. Byrnes, H_∞ control of discrete time nonlinear systems, IEEE Trans. Autom. Control 4 (1996) 494–510.

[19] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.

[20] H. Chen, C.W. Scherer, Disturbance attenuation with actuator constraints by moving horizon H_∞ control, in: Proceedings of IFAC ADCHEM'03, Hong Kong, China, 2003, pp. 447–452.

[21] H. Chen, X.-Q. Gao, H. Wang, An improved moving horizon H_∞ control scheme through Lagrange duality, Int. J. Control 79 (2006) 239–248.

[22] B. Kouvaritakis, J.A. Rossiter, J. Schuurmans, Efficient robust predictive control, IEEE Trans. Autom. Control 45 (2000) 1545–1549.

[23] M.V. Kothare, V. Balakrishnan, M. Morari, Robust constrained model predictive control using linear matrix inequalities, Automatica 32 (1996) 1361–1379.

[24] L. Magni, G.D. Nicolao, R. Scattolini, A stabilizing model-based predictive control algorithm for nonlinear systems, Automatica 37 (2001) 1351–1362.

[25] D. Limon, J.B. Brovo, T. Alamo, E.F. Camacho, Robust MPC of constrained nonlinear systems based on interval arithmetic, IEEE Proc. Control Theory Appl. 152 (2005) 325–332.

[26] Y.Y. Cao, P.M. Frank, Analysis and synthesis of nonlinear time-delay systems via fuzzy control approach, IEEE Trans. Fuzzy Syst. 8 (2000) 200–211.

[27] H. Chen, K.-H. Guo, Constrained H_∞ control of active suspensions: an LMI approach, IEEE Trans. Control Syst. Technol. 13 (2005) 412–421.

[28] V. Misra, W.B. Gong, D. Towsley, Fluid-based analysis of a network of AQM routers supporting TCP flows with an application to RED, in: Proceedings of ACM/SIGCOMM, 2000.

[29] F. Zheng, J. Nelson, An H_∞ approach to the controller design of AQM routers supporting TCP flows, Automatica 45 (2009) 757–763.

[30] K.B. Kim, Design of feedback controls supporting TCP based on the state-space approach, IEEE Trans. Autom. Control 7 (2006) 1086–1099.

[31] M.C.M. Teixeira, S.H. Zak, Stabilizing controller design for uncertain nonlinear systems using fuzzy models, IEEE Trans. Fuzzy Syst. 7 (1999) 133–142.

[32] H. Chen, F. Xu, Y. Xi, FPGA/SOPC based implementation of model predictive controller, Control Theory Appl. http://dx.doi.org/10.1049/iet-cta.2010.0443, in press.

[33] Y. Wang, S. Boyd, Fast model predictive control using online optimization, IEEE Trans. Control Syst. Technol. 18 (2010) 267–278.